LSC-O
Learning Center

Completing the Square

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The method of "completing the square" offers an option for solving quadratic equations that are not factorable with integers alone (solutions may include fractions, radicals, or imaginary numbers).

Step 1: *Rearrange–Divide (as needed)*

- Rearrange the equation, placing the constant term to the right of the equal sign and the variable terms to the left. ***Leave blanks on each side of the equation for values you will add in the next step.***
- If necessary, divide both sides of the equation by the coefficient of the highest power term to make the leading coefficient 1. ***Completing the square won't work unless the lead coefficient is 1!***

Example 1	Example 2
$x^2 - 1 = -5x$ $x^2 - 1 + 1 + 5x = -5x + 5x + 1$ $x^2 + 5x + \_\_ = 1 + \_\_$	$2x^2 - 6x + 20 = 0$ $\frac{2x^2}{2} - \frac{6x}{2} + \_\_ = -\frac{20}{2} + \_\_$ $x^2 - 3x + \_\_ = -10 + \_\_$

Step 2: *Half–Square–Add*

- Take ½ (divide by 2) the coefficient of x; then square the result.
- Add that number to both sides of the equation.

$coefficient\ of\ x\ is\ 5 \rightarrow \frac{1}{2}(5) = \frac{5}{2}$ $\left(\frac{5}{2}\right)^2 = \frac{25}{4}$ $x^2 + 5x + \frac{25}{4} = 1 + \frac{25}{4}$	$coefficient\ of\ x\ is -3 \rightarrow \frac{1}{2}(-3) = -\frac{3}{2}$ $\left(-\frac{3}{2}\right)^2 = \frac{9}{4}$ $x^2 - 3x + \frac{9}{4} = -10 + \frac{9}{4}$

Step 3: *Factor Left–Simplify Right*

- Factoring the left side will result in two identical binomials which can be written as a perfect square. ***This is the square you completed!***
- Simplify the right side by adding the constant and number that resulted from step 2.

$\left(x + \frac{5}{2}\right)\left(x + \frac{5}{2}\right) = \frac{4}{4} + \frac{25}{4}$ $\left(x + \frac{5}{2}\right)^2 = \frac{29}{4}$	$\left(x - \frac{3}{2}\right)\left(x - \frac{3}{2}\right) = -\frac{40}{4} + \frac{9}{4}$ $\left(x - \frac{3}{2}\right)^2 = -\frac{31}{4}$

Step 4: *Solve!*

- Use the **square root property** (take the square root of both sides) to solve for x.
- Remember to use both **positive** and **negative** values on the right to allow for two solutions.

* $i\ denotes\ \sqrt{-1}$ (imaginary number)

$\sqrt{\left(x + \frac{5}{2}\right)^2} = \pm\sqrt{\frac{29}{4}} \rightarrow \left(x + \frac{5}{2}\right) = \pm\frac{\sqrt{29}}{2}$ $x = -\frac{5}{2} \pm \frac{\sqrt{29}}{2}$ $solution\ set\ is \left\{\frac{-5 + \sqrt{29}}{2}, \frac{-5 - \sqrt{29}}{2}\right\}$	$\sqrt{\left(x - \frac{3}{2}\right)^2} = \pm\sqrt{-\frac{31}{4}} \rightarrow \left(x - \frac{3}{2}\right) = \pm\frac{i\sqrt{31}}{2}$ * $x = \frac{3}{2} \pm \frac{i\sqrt{31}}{2}$ $solution\ set\ is \left\{\frac{3 + i\sqrt{31}}{2}, \frac{3 - i\sqrt{31}}{2}\right\}$

See next side of this sheet for more about completing the square.

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More Ways to Use Completing the Square

Besides providing an alternative to factoring for solving quadratic equations, "completing the square" is useful for writing equations in specific formats.

Standard Form of a Circle

$$(x-h)^2+(y-k)^2=r^2$$

Given:

$$4x^2+4y^2-5x+8y-2=0$$

Step 1: ***Divide & Group, Move Constant Rt.***

- If needed, divide to make the coefficients $\mathbf{x^2}$ and $\mathbf{y^2}$ one (these coefficients are always identical in circle equations).
- Group **x terms** and **y terms**, leaving blanks for the values to be added in the next step.
- Move the constant to the right of the equal sign; leave two blanks for values you will add.

$$\frac{4x^2}{4}+\frac{4y^2}{4}-\frac{5x}{4}+\frac{8y}{4}-\frac{2}{4}=\frac{0}{4}$$

$$x^2+y^2-\frac{5}{4}x+2y-\frac{1}{2}=0$$

$$x^2-\frac{5}{4}x+\_\_+y^2+2y+\_\_=\frac{1}{2}+\_\_+\_\_$$

Step 2: ***Complete the Square Twice, (Add)***

- Complete the square for the **x group** and for the **y group** (***Half-Square-Add, Factor***).
- Values added to the left side must also be added to the right (***remember to use a common denominator when adding fractions***).

$$\frac{1}{2}\left(-\frac{5}{4}\right)=-\frac{5}{8} \qquad \frac{1}{2}(2)=\frac{1}{2}\left(\frac{2}{1}\right)=\frac{2}{2}=1$$

$$\left(-\frac{5}{8}\right)^2=\frac{25}{64} \qquad (1)^2=1$$

$$x^2-\frac{5}{4}+\frac{25}{64}+y^2+2y+1=\frac{1}{2}+\frac{25}{64}+1=\frac{32+25+64}{64}$$

This is the standard form of a circle!

$$\left(x-\frac{5}{8}\right)^2+(y+1)^2=\frac{121}{64}$$

Step 3: ***Identify Center and Radius***

- Change signs if needed to conform to the standard circle equation. The **center** is **(h, k)**.
- The **radius,** $r=\sqrt{r^2}$.

$$\left(x-\frac{5}{8}\right)^2+\left(y-(-1)\right)^2=\frac{121}{64}$$

$$(h,k)=\left(\frac{5}{8},-1\right);\ r=\sqrt{\frac{121}{64}}=\frac{11}{8}$$

Vertex form of a Quadratic Function (parabola)

$$g(x)=a(x-h)^2+k$$

Given:

$$g(x)=-2x^2-4x+3$$

Step 1: ***Group & Factor***

- Group x terms within parentheses.
- If the coefficient of $\mathbf{x^2}$ is anything other than one, factor out that number from the **x term group**. Leave a blank for the value to add in the next step.
- The constant remains outside the parentheses. To balance the equation, you must subtract whatever you add, so leave a blank behind the constant.

$$g(x)=(-2x^2-4x)+3$$

$$g(x)=\left(\frac{-2x^2}{-2}-\frac{4x}{-2}\right)+3$$

$$g(x)=-2(x^2+2x+\_\_\ )+3-\_\_$$

Step 2: ***Complete the Square, (Add-Mult.-Subtract)***

- Complete the square for the **x group** (***Half-Square-Add, Factor***).
- ***Multiply*** the number that completes the square by the number you factored out; ***subtract*** this number from the constant.

$$\frac{1}{2}(2)=1,\ (1)^2=1\ (add\ within\ parentheses)$$

$$-2(1)=-2\ (subtract\ outside\ parentheses)$$

By the distributive property, you really add -2, so you must subtract the same number to balance the equation.

$$g(x)=-2(x^2+2x+1)+3-(-2)$$

This is the vertex form of a quadratic function!

$$g(x)=-2(x+1)^2+5$$

Step 3: ***Identify the Vertex***

- Change the sign in the parentheses if needed to conform to the standard parabola equation. The **vertex** is **(h, k)**.

$$g(x)=-2\left(x-(-1)\right)^2+5$$

$$(h,k)=(-1,5)$$

Search for "completing the square" at PurpleMath.com for additional examples and applications.

Find more helpful math resources at the Learning Center Math Tutorials page: http://www.lsco.edu/learningcenter/math.asp

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