Factoring Polynomials

Factoring *a polynomial is the process of writing it as the product of two or more polynomial factors.*

Example: $7x^2 + 35x + 42 = 7(x+2)(x+3)$ — *one monomial factor* (7) *and two binomial factors* $(x+2)$ *and* $(x+3)$

Set the factors of a polynomial equation (as opposed to an expression) equal to zero in order to solve for a variable: ***Example:*** *To solve* $7x^2 + 35x + 42 = 0 \rightarrow x + 2 = 0,\ x = -2;$ *and* $x + 3 = 0,\ x = -3$

The flowchart below illustrates a sequence of steps for factoring polynomials.

First, always factor out the *Greatest Common Factor* (GCF), if one exists.

Is the equation a *Binomial* or a *Trinomial*?

Yes → **Special Cases?**

Binomial (two terms)

1. **Difference of Squares:** $a^2 - b^2 = (a-b)(a+b)$
2. **Sum of Squares:**[1,3] $a^2 + b^2 = Prime$
3. **Difference of Cubes:**[1,2] $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
4. **Sum of Cubes:**[1,2] $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

Trinomial (three terms)

Perfect Square Trinomial:

1. $x^2 + 2xy + y^2 = (x+y)^2$
2. $x^2 - 2xy + y^2 = (x-y)^2$

No Special Cases

Choose:

1. **Factor by Grouping**
2. **Complete the Square**[4]
3. **Use the Quadratic Formula**[4]

No → **Four or more terms**

Factor by Grouping:

1. **Group the terms with common factors and factor out the GCF from each grouping.**
2. **Continue factoring—by looking for Special Cases, Grouping, etc.—until the equation is in simplest form (or all factors are Prime).**

[1] ***Prime*** *polynomials cannot be factored using integers alone.*

The ***Sum of Squares*** *and the quadratic factors of the* ***Sum and Difference of Squares*** *are always* ***Prime****.*

[2] *Use* ***S.O.A.P*** *to remember the signs for the factors of the* ***Sum*** *and* ***Difference of Cubes****:*
S*ame,*
O*pposite,*
A*lways* ***P****ositive*

[3] ***FYI:*** *A* ***Sum of Squares*** *can be factored using* ***imaginary numbers*** *if you rewrite it as a* ***Difference of Squares****:* $a^2 + b^2 = [a^2 - (-b^2)] = [a - (b\sqrt{-1})][a + (b\sqrt{-1})] = (a - bi)(a + bi)$

[4] ***Completing the Square*** *and the* ***Quadratic Formula*** *are primarily methods for solving equations rather than simply factoring expressions.*

Also, if the ***GCF*** *doesn't contain a variable, it may not be necessary to factor it out prior to using either of these methods. However, doing so will provide smaller coefficients to work with.*

This process is applied in the following examples

Factoring steps and most examples are adapted from Professor Elias Juridini, Lamar State College-Orange.

Factoring Examples

Binomials:

1. $6x^2 + 12x$: First, divide each term by the **GCF** to get the **quotient** $\rightarrow \frac{6x^2}{6x} + \frac{12x}{6x} = x + 2$.

Then, show the **quotient** multiplied by the **GCF** $\rightarrow 6x(x+2)$.

No special cases apply to the binomial **quotient** $(x+2)$, so the factors are $6x$ and $(x+2)$.

Factored: $6x^2 + 12x = 6x(x+2)$

---

2. $x^4 - 16$: This polynomial expression has no **GCF** (*other than* **1** *and* **−1**).

However, it can be expressed as a **Difference of Squares**: $a^2 - b^2 \rightarrow (x^2)^2 - 4^2$.

Use the square root of each term, $\sqrt{x^4} = \sqrt{(x^2)^2} = x^2$ and $\sqrt{16} = \sqrt{4^2} = 4$ to fill in the formula:

$$\textit{If } a = x^2 \textit{ and } b = 4, \textit{then} \ \ldots$$

$$a^2 - b^2 = (a - b)(a + b)$$

$$(x^2)^2 - 4^2 = (x^2 - 4)(x^2 + 4)$$

Continue to factor another **Difference of Squares**: $(x^2 - 4) = (x - 2)(x + 2)$.

The next factor $(x^2 + 4)$ is a **Sum of Squares**, which is **Prime**.

Factored: $x^4 - 16 = (x-2)(x+2)(x^2+4)$

---

3. $729x^3 - 1$: There's no **GCF** to factor out, but you should recognize this polynomial expression as a **Difference of Cubes**: $a^3 - b^3 \rightarrow 9^3x^3 - 1^3$.

Use the cube root of each term, $\sqrt[3]{729x^3} = \sqrt[3]{9^3x^3} = 9x$ *and* $\sqrt[3]{1} = \sqrt[3]{1^3} = 1$ to fill in the formula:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

- In the first set of parentheses, place each term to the first power: $(9x - 1)$.
- In the second set, square the first and the last term: $[(9x)^2 + ? + 1^2]$
  - o The middle term is the product of the first and the last: $[(9x)^2 + 9x(1) + 1^2]$
  - o ****Note that this quadratic factor is Prime!***
- Use **S.O.A.P** to remember the signs: **S**ame in first set of parentheses; **O**pposite, followed by **A**lways **P**ositive in second set of parentheses.

$$\text{If } a = 9x \text{ and } b = 1, \text{then} \ \ldots$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

$$9^3x^3 - 1^3 = (9x - 1)[9^2x^2 + 9x(1) + 1^2]$$

Factored: $729x^3 - 1 = (9x - 1)(81x^2 + 9x + 1)$.

Trinomials:

1. $x^2 + 4x + 4$: There's no **GCF** to factor out of this expression, so check for a **Perfect Square Trinomial**—

$$x^2 + 2xy + y^2 = (x+y)^2 \text{ or } x^2 - 2xy + y^2 = (x-y)^2$$

ASK: •Are the first and the last terms perfect squares? x^2 and $4 = 2^2$ — **Yes**!

•Is the middle term two times the product of the square roots of the first and last terms?

First Term $\boldsymbol{x^2} \rightarrow \sqrt{x^2} = \boldsymbol{x}$, Last Term $\mathbf{4} \rightarrow \sqrt{4} = \mathbf{2}$, and $\mathbf{2(2x) = 4x}$ — **Yes**!

If "**Yes**" to both of the above:

•Place the square roots of the first and last terms in parentheses: $(x\ \ ?\ \ 2)$

•Use the sign of the middle term: $(x + 2)$

•Square the whole thing: $(x + 2)^2$

$$\text{If } \sqrt{x^2} = x \text{ and } \sqrt{4} = 2\text{, then } \ldots$$

$$\boldsymbol{x^2 + 2xy + y^2 = (x + y)^2}$$

$$\boldsymbol{x^2 + 4x + 4} \rightarrow \quad \boldsymbol{x^2 + 2(2x) + 2^2 = (x + 2)^2}$$

Factored: $\boldsymbol{x^2 + 4x + 4 = (x + 2)^2}$

---

2. $\boldsymbol{4x^2 + 2x - 20 = 0}$: Factor out the **GFC, 2**, in this equation to get $\boldsymbol{2(2x^2 + x - 10)}$.

Be sure the trinomial factor isn't a special case— it isn't— and select a factoring method.

a. Try **Grouping**:

- First, multiply the **coefficient of the first** term, 2, and the constant, 10: **2•10=20**.
- Then find the factors of this product, **20**, that add/subtract to yield the coefficient of the middle term, which is **1**.
  Factors of **20**: **1•20**, **2•10**, and **4•5**. Of these factors, **4** and **5** will subtract to give **1**.
  Because the middle term is positive, we use **+5** and **–4**.
- Replace the middle term with **–4** and **+5:** $\boldsymbol{2x^2 - 4x + 5x - 10}$.
  - o It's important to order the two new middle terms so a **GCF** can be factored from each pair: $\boldsymbol{2x^2 - 4x + 5x - 10}$, but not $\boldsymbol{2x^2 + 5x - 4x - 10}$.
- Group the first and last pairs of terms; factor out a **GCF** from each, and rewrite the problem:
  $(2x^2 - 4X)$ *and* $(5x - 10)$ $\rightarrow$ $2x(x - 2)$ *and* $5(x - 2)$ $\rightarrow$ $\boldsymbol{2x(x - 2) + 5(x - 2)}$.
  - o Note that $(\boldsymbol{x - 2})$ appears twice. If this doesn't happen, reorder your middle terms.
- Factor out the **GCF**—which is $(\boldsymbol{x - 2})$—and rewrite in factored form.
  $$\boldsymbol{2x(x - 2) + 5(x - 2) \quad \rightarrow \quad (x - 2)(2x + 5)}$$
- Remember to include the ***2*** factored out at the beginning when you write the whole equation in factored form

Factored: $\boldsymbol{4x^2 + 2x - 20 = 0 \ \rightarrow\ 2(x - 2)(2x + 5) = 0}$

- Recall that this example is an ***equation*** set equal to zero (not simply an ***expression***). That means we can solve for ***x*** by setting each factor containing a variable equal to zero.

$$\boldsymbol{x - 2 = 0} \text{ or } \boldsymbol{2x + 5 = 0}$$

$$\boldsymbol{x - 2 + 2 = 0 + 2 = 2} \text{ or } \boldsymbol{2x + 5 - 5 = 0 - 5 \rightarrow \frac{2x}{2} = -\frac{5}{2}}$$

Solutions: $\boldsymbol{x = 2}$ or $\boldsymbol{x = -\frac{5}{2}}$

**Always check by plugging your answers into the original equation to verify them!*

---

b. Now let's **solve** the same equation, $\boldsymbol{4x^2 + 2x - 20 = 0}$, by **Completing the Square**:

- Make sure the equation is in the **General** (quadratic) **Form**— $\boldsymbol{ax^2 + bx + c = 0}$—and it is.
- The **GCF** is 2, but you don't need to factor it out—doing so will not affect the solutions.

( ) Factor by Grouping ( )

Completing the Square

- Divide each term by the first term's coefficient (unless the first coefficient is already 1):

$$\frac{4x^2+2x-20}{4}=\frac{4x^2}{4}+\frac{2x}{4}-\frac{20}{4}=\boldsymbol{x^2+\frac{1}{2}x-5}$$

- Isolate the constant on the right side of the equal sign. Then take ½ the middle term's coefficient (multiply by ½ or divide by 2), **square it**, and **add this number to both sides** of the equation:

$$x^2+\frac{1}{2}x-5+5=0+5\rightarrow \boldsymbol{x^2+\frac{1}{2}x=5}$$

The **Middle Term** is $\frac{1}{2}$ → Take $\frac{1}{2}$ of $\frac{1}{2}$ and square it: $\left[\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\right]^2\rightarrow\left(\frac{1}{4}\right)^2\rightarrow\frac{1^2}{4^2}\rightarrow\boldsymbol{\frac{1}{16}}$

Add $\frac{1}{16}$ to both sides: $x^2+\frac{1}{2}x+\frac{1}{16}=5+\frac{1}{16}$

Use a common denominator to add fractions: $\left(\frac{5}{1}\right)\left(\frac{16}{16}\right)+\frac{1}{16}=\frac{80}{16}+\frac{1}{16}=\boldsymbol{\frac{81}{16}}$

- Now we have $\boldsymbol{x^2+\frac{1}{2}x+\frac{1}{16}=\frac{81}{16}}$, which is a Perfect Square Trinomial on the left.

**The point of Completing the Square is to create a Perfect Square Trinomial!*

Compare with the formula: $\boldsymbol{x^2+2xy+y^2=(x+y)^2}$

In $\boldsymbol{x^2+\frac{1}{2}x+\frac{1}{16}}$, $\boldsymbol{x}=\sqrt{\boldsymbol{x^2}}=\boldsymbol{x}$ and $\boldsymbol{y}=\sqrt{\frac{1}{16}}=\frac{1}{4}$ → The middle term is $\boldsymbol{2\left[(x)\left(\frac{1}{4}\right)\right]=\frac{1}{2}x}$

The trinomial can be written as $\boldsymbol{x^2+2\left[(x)\left(\frac{1}{4}\right)\right]+\left(\frac{1}{4}\right)^2}$, so it can be factored as $\boldsymbol{\left(x+\frac{1}{4}\right)^2}$

- Rewrite the equation with the left side factored:

$$\boldsymbol{\left(x+\frac{1}{4}\right)^2=\frac{81}{16}}$$

- The final step is to **extract the roots** and **solve for x**:

$$\sqrt{\left(x+\frac{1}{4}\right)^2}=\pm\sqrt{\frac{81}{16}}\rightarrow x+\frac{1}{4}=\pm\frac{9}{4}\rightarrow x+\frac{1}{4}-\frac{1}{4}=\pm\frac{9}{4}-\frac{1}{4}$$

$$x=\frac{9}{4}-\frac{1}{4}=\frac{8}{4}=2\quad\text{or}\quad x=-\frac{9}{4}-\frac{1}{4}=-\frac{10}{4}=-\frac{5}{2}$$

Solutions: $\boldsymbol{x=2}$ or $\boldsymbol{x=-\frac{5}{2}}$

**The solutions are the same as obtained from the previous method!*

---

c. Finally, let's try using the Quadratic Formula to solve the equation $\boldsymbol{4x^2+2x-20=0}$:

Quadratic Formula

$$\boldsymbol{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$$

- Factor out the **GCF** to get $\boldsymbol{2(2x^2+x-10)=0}$.

- Note that factoring out the **2** will not affect the solutions, but it does result in **smaller coefficients** to plug into the **Quadratic Formula.**
- Make sure the equation is in **General** (quadratic) **Form**, $\boldsymbol{ax^2 + bx + c = 0}$, and it is.
- Identify ***a***, ***b***, and ***c*** in $\boldsymbol{2x^2 + x - 10 = 0}$: $\boldsymbol{a} = 2$, $\boldsymbol{b} = 1$, $\boldsymbol{c} = -10$
- Plug these numbers into the formula and solve for ***x***:

$$x = \frac{-1 \pm \sqrt{1^2 - 4(2)(-10)}}{2(2)}$$

$$x = \frac{-1 \pm \sqrt{1 + 80}}{4} \rightarrow \frac{-1 \pm \sqrt{81}}{4} \rightarrow \frac{-1 \pm 9}{4} \rightarrow$$

$$x = \frac{-1 + 9}{4} = \frac{8}{4} = 2 \quad \text{or} \quad x = \frac{-1 - 9}{4} = -\frac{10}{4} = -\frac{5}{2}$$

$$\boxed{\text{Solutions: } x = 2 \quad \text{or} \quad x = -\frac{5}{2}}$$

Again, we obtain the same solutions as we did by other methods!

More Than Four Terms:

1. $\boldsymbol{5x^2 - 60y - 10y^2 + 10x}$: Factor the expression by **Grouping** and continue factoring until completely simplified**.**

- Factor out the **GCF**, **5**, to get $\boldsymbol{5(x^2 - 12y - 2y^2 + 2x)}$ and then **group the terms** so that another **GCF** can be removed from each grouping.

$$5(x^2 + 2x - 2y^2 - 12y) \rightarrow 5[x(x + 2) - 2y(y + 6)]$$

- We are unable to obtain a pair of identical binomial factors (as in the earlier Grouping example). Therefore, no further factoring is possible and we see that the expression has only two factors, ***5*** and $[\boldsymbol{x(x + 2) - 2y(y + 6)}]$**.**

$$\boxed{\text{Factored: } 5x^2 - 60y - 10y^2 + 10x = 5[x(x + 2) - 2y(y + 6)]}$$

---

2. $\boldsymbol{x^2 - y^2 + 6x + 9}$: The terms of this expression appear to need no rearrangement. The first pair, $\boldsymbol{x^2 - y^2}$, are a **Difference of Squares** and the second pair, $\boldsymbol{6x + 9}$, have a **GCF** of **3:**

- Factor each pair: $\boldsymbol{x^2 - y^2 + 6x + 9 = (x - y)(x + y) + 3(2x + 3)}$

  However, there is no **GCF** to factor out of these pairs of factors. The expression remains the **sum of polynomial products** rather than the **product of two or more polynomial factors** (the definition of the factored form).

- When one way of grouping doesn't work, try another . . .

  Rearrange the terms to form a **Perfect Square Trinomial** and a **Difference of Squares**:

$$x^2 - y^2 + 6x + 9 = x^2 + 6x + 9 - y^2 = (x + 3)^2 - y^2$$

- Factor the **Difference of Squares**:

$$(x + 3)^2 - y^2 = [(x + 3) - y][(x + 3) + y] = (x + 3 - y)(x + 3 + y)$$

$$\boxed{\text{Factored: } x^2 - y^2 + 6x + 9 = (x + 3 - y)(x + 3 + y)}$$

This example is adapted from Gustafson & Fisk, *College Algebra*, 8th ed., Thompson Learning, Inc. 2004, p. 55.