How to Solve a System of Equations using the . . .

MATRIX

Using the nine **coefficients** and the three **constants** in the following system of equations . . .

$$\begin{cases} x + 2y + z = 8 \\ 2x + y - z = 1 \\ x + y - 2z = -3 \end{cases}$$

End Result →

Work through a series of **Row Operations** to create a **matrix** of the following form:

$$\left[\begin{array}{ccc|c} +1 & +0 & +0 & \pm ? \\ +0 & +1 & +0 & \pm ? \\ +0 & +0 & +1 & \pm ? \end{array}\right] \begin{array}{l} \leftarrow x \\ \leftarrow y \\ \leftarrow z \end{array}$$

The **final column** will contain the **three solutions.**

The Format: Begin with all equations in **STANDARD FROM** (**Ax + By + Cz= D**). Then, express the system as a matrix, called the **ARGUMENT MATRIX**.

$$\begin{cases} x + 2y + z = 8 \\ 2x + y - z = 1 \\ x + y - 2z = -3 \end{cases}$$

ARGRUMENT MATRIX →

$$\left[\begin{array}{ccc|c} +1 & +2 & +1 & +8 \\ +2 & +1 & -1 & +1 \\ +1 & +1 & -2 & -3 \end{array}\right]$$

Row 1 – R_1
Row 2 – R_2
Row 3 – R_3

The Rules: Consider the three basic **ROW OPERATIONS** (or mathematically "legal" procedures) you can use to achieve a matrix with a **diagonal of 1s,** and 0s for all other coefficients:

I. ***SWITCH*** **— Interchange any two rows.**

II. ***MULTIPLY (or DIVIDE)*** **the elements of any row by a nonzero number.**

III. ***ADD & REPLACE*** **— Add a multiple of one row to a different row; then replace one of these rows with the result.**

The Process: Although knowledge of formal methods for solving matrices is helpful (see Gaussian and Gauss-Jordan Elimination in "Solving Systems of Equations" at *Paul's Online Math Notes*, Lamar University), an intuitive application of row operations—as we demonstrate here—is often sufficient for simpler systems. In any case, working with matrices can be a tedious, error-prone process, so **WRITE OUT ALL STEPS!** ****Note that there may be more than one sequence of steps to solve a given matrix.***

Phase 1: Since R_1 begins with a **1**, our diagonal is already started. To get **0s** at the beginning of R_2 and R_3, we can use **Row Operation III** twice.

$$(-2)\bullet R_1 + R_2 \rightarrow R_2: [(-2)(1) + 2 = 0 \quad (-2)(2) + 1 = -3 \quad (-2)(1) + -1 = -3 \quad (-2)(8) + 1 = -15]$$

$$\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \begin{bmatrix} +1 & +2 & +1 & +8 \\ +2 & +1 & -1 & +1 \\ +1 & +1 & -2 & -3 \end{bmatrix} \xrightarrow{-2R_1 + R_2} \begin{bmatrix} +1 & +2 & +1 & +8 \\ 0 & -3 & -3 & -15 \\ +1 & +1 & -2 & -3 \end{bmatrix}$$

R_1 and R_3 do not change.

(-1)•R_1 +R_3→ R_3:[(-1)(1) + 1 = 0 (-1)(2) + 1 = -1 (-1)(1) + -2 = -3 (-1)(8) + -3 = -11]

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \begin{bmatrix} +1 & +2 & +1 & +8 \\ 0 & -3 & -3 & -15 \\ +1 & +1 & -2 & -3 \end{bmatrix} \xrightarrow{-1R_1+R_3} \begin{bmatrix} +1 & +2 & +1 & +8 \\ 0 & -3 & -3 & -15 \\ 0 & -1 & -3 & -11 \end{bmatrix}$$

R_1 and R_2 do not change.

Phase 2: To get the next **1** in our diagonal, we use **Row Operation II** and multiply R_2 by **-1/3** (or divide by **-3**).

(-1/3)•R_2 → R_2 : [(-1/3)(0) = 0 (-1/3)(-3) = 1 (-1/3)(-3) = 1 (-1/3)(-15) = 5]

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \begin{bmatrix} +1 & +2 & +1 & +8 \\ 0 & -3 & -3 & -15 \\ 0 & -1 & -3 & -11 \end{bmatrix} \xrightarrow{-\frac{1}{3}R_2} \begin{bmatrix} +1 & +2 & +1 & +8 \\ 0 & +1 & +1 & +5 \\ 0 & -1 & -3 & -11 \end{bmatrix}$$

R_1 and R_3 do not change.

Phase 3: We can get a couple more **0s** (in R_1 and R_3) by using **Row Operation III** again.

(-2)•R_2 +R_1 → R_1 : [(-2)(0) + 1 = 1 (-2)(1) + 2 = 0 (-2)(1) + 1 = -1 (-2)(5) + 8 = -2]

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \begin{bmatrix} +1 & +2 & +1 & +8 \\ 0 & +1 & +1 & +5 \\ 0 & -1 & -3 & -11 \end{bmatrix} \xrightarrow{-2R_2+R_1} \begin{bmatrix} +1 & 0 & -1 & -2 \\ 0 & +1 & +1 & +5 \\ 0 & -1 & -3 & -11 \end{bmatrix}$$

R_2 and R_3 do not change.

This time, we can skip the multiplication—just add R_2 and R_3 and replace.

R_2 +R_3 → R_3 : [0 + 0 = 0 1 + -1 = 0 1 + -3 = -2 5 + -11 = -6]

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \begin{bmatrix} +1 & 0 & -1 & -2 \\ 0 & +1 & +1 & +5 \\ 0 & -1 & -3 & -11 \end{bmatrix} \xrightarrow{R_2+R_3} \begin{bmatrix} +1 & 0 & -1 & -2 \\ 0 & +1 & +1 & +5 \\ 0 & 0 & -2 & -6 \end{bmatrix}$$

R_1 and R_2 do not change.

Phase 4: To get the last **1** we need to finish our diagonal, we use **Row Operation II** and multiply R_3 by -**1/2** (or divide by **-2**).

(-1/2)•R_3 → R_3 : [(-1/2)(0) = 0 (-1/2)(0) = 0 (-1/2)(-2) = 1 (-1/2)(-6) = 3]

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \begin{bmatrix} +1 & 0 & -1 & -2 \\ 0 & +1 & +1 & +5 \\ 0 & 0 & -2 & -6 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_3} \begin{bmatrix} +1 & 0 & -1 & -2 \\ 0 & +1 & +1 & +5 \\ 0 & 0 & +1 & +3 \end{bmatrix}$$

R_1 and R_2 do not change.

Phase 5: We can get the last two **0s** we need (in R_1 and R_2) by applying **Row Operation III** once again.

$R_3 + R_1$ → R_1 : [0 + 1 = 1 0 + 0 = 0 1 + -1 = 0 3 + -2 = 1]

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \begin{bmatrix} +1 & 0 & -1 & -2 \\ 0 & +1 & +1 & +5 \\ 0 & 0 & +1 & +3 \end{bmatrix} \xrightarrow{R_3+R_1} \begin{bmatrix} +1 & 0 & 0 & +1 \\ 0 & +1 & +1 & +5 \\ 0 & 0 & +1 & +3 \end{bmatrix}$$

R_2 and R_3 do not change.

(-1)•$R_3 + R_2$ → R_2 : [(-1)(0) + 0 = 0 (-1)(0) + 1 = 1 (-1)(1) + 1 = 0 (-1)(3) + 5 = 2]

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \begin{bmatrix} +1 & 0 & 0 & +1 \\ 0 & +1 & +1 & +5 \\ 0 & 0 & +1 & +3 \end{bmatrix} \xrightarrow{-1R_3+R_2} \begin{bmatrix} +1 & 0 & 0 & +1 \\ 0 & +1 & 0 & +2 \\ 0 & 0 & +1 & +3 \end{bmatrix}$$

R_1 and R_3 do not change.

Solved! 

The solutions appear in the **fourth column**: ***X = 1, Y = 2, and Z = 3***

$$\left[\begin{array}{ccc:c} +1 & +0 & +0 & +1 \\ +0 & +1 & +0 & +2 \\ +0 & +0 & +1 & +3 \end{array}\right] \begin{matrix} = X \\ = Y \\ = Z \end{matrix}$$

Don't forget to check your solutions . . .

$$\begin{cases} x + 2y + z = 8 \\ 2x + y - z = 1 \\ x + y - 2z = -3 \end{cases} \qquad \begin{cases} 1 + 2(2) + 3 = 1 + 4 + 3 = 5 + 3 = 8 \quad \surd \\ 2(1) + 2 - 3 = 2 + 2 - 3 = 4 - 3 = 1 \quad \surd \\ 1 + 2 - 2(3) = 1 + 2 - 6 = 3 - 6 = -3 \quad \surd \end{cases}$$

If working with matrices looks like a lot of fun, you may enjoy the video "Lecture 2: Elimination with Matrices," which is part of a Linear Algebra course at the MIT Open Courseware website. 

Those who find this process a bit daunting may prefer to tackle matrices with a calculator.
See tutorial #12 under Matrix Operations—"Solving a System of Equations Using an Argument Matrix and Row Reduction"—at calulator.maconstate.edu.